THE GEOMETRY OF EXTRAORDINARY REFRACTION*

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(*Received for publication, May 21, 1948*)

ABSTRACT An expression has been deduced connecting the angle of refraction of the extraordinary ray (and of the extraordinary wave normal) in a uni-axial crystal with the angle of incidence and the inclination of the optic axis to the refracting surface. The usual special cases given in text books follow from the general expression. The peculiarities in the behaviour of the extraordinary ray at normal incidence but with different inclinations of the optic axis have been derived. Mention has also been made of peculiarities at oblique incidence, the details and data of the general case being left over to another paper.

INTRODUCTION

The only method of finding the direction of the extraordinary ray, or of the extra-ordinary wave normal, seems to be with the help of Huygens' construction. An actual geometrical construction in every case, however, cannot be regarded as a really practical method of doing so ; nor can it give quantitatively correct results. It is, therefore, necessary to find some convenient relationship between the angle of incidence and that of the extra-ordinary refraction, *i. e.* the e-ray or the e-wave normal ; some readily applicable expression like Snell's law which holds for the ordinary refraction, namely, that the ratio of the sine of the angle of incidence to that of refraction is a constant, being equal to the ordinary refractive index, μ_0.

Relationships for the extraordinary ray in uni-axial crystals in some simple and special cases, namely, the optic axis in the plane of incidence and either (*a*) parallel or (*b*) perpendicular to the refracting surface of the crystal, are found by making use of the pure geometrical properties of an ellipse and are given in all text books dealing with the matter, in connection with the experimental verification of Huygens' construction. In some books, the case of normal incidence, but any inclination of the optic axis, has also been derived, again with the help of pure geometry. The general case, when the optic axis is in any direction whatever and any angle of incidence, does not appear to have been derived, at least, is not given in any of the usual text books.

It, therefore, occurred to the author to investigate the general, or rather the semi-general, case of any angle of incidence and any inclination of the optic axis, only the optic axis remaining in the plane of incidence. This has been done here with the help of co-ordinate geometry. The most general

* Communicated by Dr. P. K. Kichlu.

case of the optic axis not being confined to the plane of incidence involves the use of solid geometry and is beyond the scope of the present investigation. The very special case when the optic axis is perpendicular to the plane of incidence does not require any special treatment, for in this case the e-ray obeys all the laws of refraction and therefore its direction is immediately determinable by the application of Snell's law again, the constant being now the extra-ordinary refractive index, μ_e.

From what has preceded, it must be obvious that this paper will deal only with uni-axial crystals. The general case of bi-axials does not fall within our purview. For the sections of the wave surface in a biaxial crystal are not simple curves and the results obtained in this paper are based on the fact that the sections of the extraordinary wave surface in a uni-axial crystal are ellipses in all azimuths but one in which they are circles. In the case of a biaxial crystal as well, the sections of the wave surface by the three principal planes are, in each case, a circle and an ellipse. Of the two rays in these sections, therefore, the one given by the circle will behave like an ordinary ray ; the other given by the ellipse, will behave like the extraordinary ray in a uni-axial. The results obtained in this paper can, therefore, be adapted quite easily to give the direction of the extraordinary ray when the plane of incidence is parallel to one of the three principal sections of the biaxial crystal.

SYMBOLS USED

For the purpose of deriving the relationship wanted, we take the case of a negative uni-axial crystal, like calcite (calcspar), although we shall see that the results obtained in the final working form applies equally to the positive crystals.

Since the case of a uni-axial is only a particular case of a bi-axial, we shall take here that the major and minor semi-axes of the generating ellipse of the spheroid giving the extraordinary wave surface are given by a and c. For the case in hand, that of a negative crystal, the spheroid is oblate and the ordinary and extraordinary wave, or ray, velocities will be given respectively by c and a, taking the velocity of light in vacuo as unity. Thus, $\mu_o = 1/c$, $\mu_e = 1/a$. We take the two axes of reference to be the axis of x and that of z, taking the optic axis, the minor axis, to be the axis of x. Taking O as the point of incidence on the face of the crystal, a line, OX, drawn from O *into* the crystal will be taken as the positive direction of the axis of x. The line, OZ, perpendicular to OX and "above" or on the right side of OX, will be taken as the positive direction for the z-axis. The equation of the generating ellipse is, then,

$$x^2/c^2 + z^2/a^2 = 1 \qquad \ldots (1)$$

We take the plane of the paper as the plane of incidence, with the optic axis in this plane, and take α as the angle between the optic axis and the refracting face of the crystal. In Fig. 1, the angle XOT thus $=\alpha$. The angle between the refracted extraordinary ray and the optic axis is taken to be θ, that between the optic axis and the extraordinary wave normal as ϕ. The angle of incidence, *i.e.*, the angle between the incident ray and the normal to the surface at the point of incidence, which is the same as the angle between the incident (plane) wave front and the refracting surface, is, as usual, taken to be i. The angles of refraction for the ordinary ray, the extraordinary ray and the extraordinary wave normal are taken to be r, r', r'', respectively. These are the angles between the normal to the refracting surface and, respectively, the ordinary ray, the e-ray and the normal to the extraordinary refracted (plane) wave front.

We shall find θ (or ϕ) in terms of α and i. θ (or ϕ) being known, r' (or r'') become immediately determined, as is done at the end of the next section after giving the convention regarding signs.

THE SIGN CONVENTION

We shall always consider the incident ray to lie on the left of the normal to the refracting surface and consider i to be positive in this case. In ordinary refraction, the refracted ray always lies on the other side of the normal, *i.e.*, the side of the normal other than that on which the incident ray lies. If, then, the e-ray lies on the right side of the surface normal, r' will be regarded as positive ; if it lies on the left side, r' will be taken as negative. Similarly, if the optic axis lies on the right of the surface normal, α will be taken to be positive ; if it lies on the left, α will be regarded as negative. That is to say, if the straight line to represent the optic axis is drawn from the point of incidence *into* the crystal, then if it lies on the same side of the normal as the incident ray, α will be negative : if it lies on the other side, α will be positive. (As a matter of fact, since the optic axis is taken as the axis of x, the positive direction of which is as, has just been described, the positive and negative signs of α, as given here, follow automatically).

With respect to θ (or ϕ), if this comes out to be positive, it means that the extraordinarily refracted ray lies in the first quadrant, XOZ, whether α be positive or negative. If, on the other hand, θ comes out to be negative then the refracted ray will lie in the fourth quadrant $Z'OX$, if α be positive ; or, when α is negative, then in the second quadrant, ZOX'. By drawing diagrams in all the four cases, it can be verified that the angle of refraction of the e-ray, r', is given by,

$$r'=\theta-\alpha\pm 90^\circ \qquad \dots \quad (2)$$

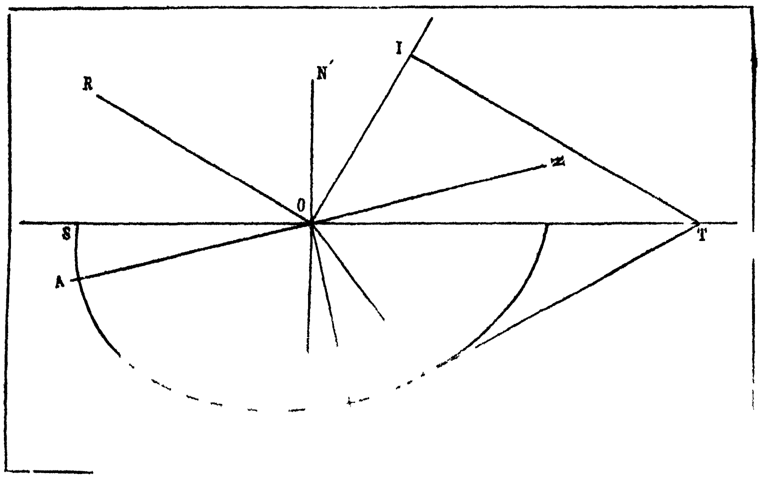
Both θ and α in the above equation are to be taken with their proper signs. The angle, 90°, is to be subtracted in only one case, when α is negative and θ positive ; in all the other three cases it is to be added.

Similarly, and with similar remarks regarding signs, for the extra-ordinary wave normal,

$$r'' = \phi - \alpha \pm 90^\circ \qquad \dots \quad (3)$$

GEOMETRICAL

Fig. 1, represents the plane of incidence. SOT is the trace of the refracting surface of the crystal, which is a plane passing through ST, perpendicular to the plane of the paper. OX is the direction of the optic axis, O being the point of incidence. NON' is the normal to the surface at O. RO is the incident ray. A straight line, OI, at right angles to RO, is the trace of the incident plane wave front, the plane of which is perpendicular to the plane of the paper. IT, at right angles to OI, is another incident ray. Taking IT=unity, and with centre O, the appropriate ellipse $SACP$ with major and minor axes $=2a$ and $2c$ respectively, is drawn so that $OA=a$ and $OC=c$. From T the tangent, TP, to the ellipse, is drawn touching it at P. Then OP is the extraordinary refracted ray corresponding to the incident



ray, RO, and the angle, $POX=\theta$, which is to be determined. The angle of refraction itself of the e-ray is $PON=r'$. The angle, RON'=the angle, IOT=the angle of incidence$=i$.

Since IT=unity, the co-ordinates of T are given by, $x=\cos\alpha/\sin i$, $z=\sin\alpha/\sin i$. Let the equation of the tangent, TP, be $z=mx+k$. Then, since TP passes through T and is tangential to the ellipse given by the equation (1), we have,

$$\frac{\sin\alpha}{\sin i} = m\frac{\cos\alpha}{\sin i} + \sqrt{c^2m^2+a^2}.$$

This gives a quadratic equation in m, namely, $m^2(\cos^2\alpha - c^2\sin^2 i) - m\sin 2\alpha + \sin^2\alpha - a^2\sin^2 i = 0$, from which we get,

$$m = \frac{\sin 2\alpha \pm 2 \sin i \sqrt{a^2\cos^2\alpha + c^2\sin^2\alpha - a^2c^2\sin^2 i}}{2(\cos^2\alpha - c^2\sin^2 i)} \qquad \ldots \quad (4)$$

This gives the two tangents that can be drawn to the ellipse from the point, T. For our present purpose we want only one of these, the "lower" one in Fig. 1, which, a little consideration will show, is given by the positive value of the radical. In what follows, therefore, we shall retain only the positive sign.

Now the point, P, satisfies the relations.

$$(a) \quad z = mx + k, \; k^2 = c^2m^2 + a^2 ;$$

$$(b) \quad x^2/c^2 + z^2/a^2 = 1.$$

P is, therefore, given by,

$$x = -mc^2/k \; ; \; z = a^2/k.$$

Therefore we have

$$\tan\theta (= z/x) = -a^2/c^2 m \qquad \ldots \quad (5)$$

where m has the value given above (equation 4) with the positive sign before the radical.

THE DIRECTION OF THE EXTRAORDINARY RAY

The direction of the extraordinary ray with respect to the optic axis, namely, θ, has been obtained above (eqn. 5). We give it its final working form by putting a and c in terms of the two refractive indices. Thus putting $c = 1/\mu_0$ and $a = 1/\mu_e$, we get that for any angle α, which the optic axis makes with the refracting surface, the relation between i, the angle of incidence, and θ, the angle which the extraordinary ray makes with the optic axis, is given by,

$$\tan\theta = \frac{-2Q(\mu_0^2\cos^2\alpha - \sin^2 i)}{P\sin^2\alpha + 2\sin i\sqrt{R}} \qquad \ldots \quad (6)$$

where $\qquad P = \mu_0\mu_e \; ; \; Q = \mu_0/\mu_e \; ;$ and

$$R = \mu_0^2\cos^2\alpha + \mu_e^2\sin^2\alpha - \sin^2 i.$$

It must be remembered that in the above equation α must be taken with the proper sign, namely, positive if the optic axis is inclined towards the right of the surface normal, negative if towards the left. It may also be recalled that we take i to be always positive, *i.e.*, the incident ray towards the left of the surface normal. This only means that in the foregoing expression (and similar expressions which follow) only sin 2α, and no other term, will be affected by the sign of α.

Since the actual angle of refraction, r', is given by $r' = \theta - \alpha \pm 90°$, so that $\theta = r' + \alpha \pm 90°$, and, therefore, $\tan\theta = -\cot(\alpha + r')$, the angle of refraction itself of the extraordinary ray is given by

$$\tan(\alpha + r') = \frac{P\sin 2\alpha + 2\sin i\sqrt{R}}{2Q(\mu_0^2\cos^2\alpha - \sin^2 i)} \qquad \ldots \quad (7)$$

It should be noted that when the right hand side of the equations, 6, 7, 8, 12, come out to be negative, care has to be exercised in selecting the correct angle. For both $\tan(-A)$ and $\tan(180°-A)=-\tan A$, where A is an acute angle.

THE EXTRAORDINARY WAVE NORMAL

Since the equation of the central perpendicular to the tangent, $z=mx+k$, is given by $z=-x/m$, where m is given by equation (4), the angle, ϕ, which the normal to the extraordinary wave front makes with the optic axis will be given by

$$\tan\phi=-\frac{2}{Q}\,\frac{\mu_0{}^2\cos^2\alpha-\sin^2 i}{P\sin 2\alpha+2\sin i\sqrt{R}},$$

where P, Q, R have the same values as given with equation (6), and, again, α is to be taken with the appropriate sign, i being always regarded as positive.

Thus the relationship between ϕ and θ is,

$$\tan\phi=\frac{1}{Q^2}\,.\,\tan\theta \qquad \dots \quad (8)$$

Or, putting θ and ϕ in terms of r' and r'',

$$\tan(\alpha+r'')=Q^2\tan(\alpha+r') \qquad \dots \quad (9)$$

It may be mentioned here that very few authors deal specifically with the direction of the extraordinary wave normal.

POSITIVE CRYSTALS

In the foregoing treatment the optic axis, the axis of x, was taken the minor axis of the generating ellipse of which the semi-axes are a and c. That gave the case of the negative crystals. In positive crystals the optic axis is the major axis of the ellipse. Taking this now as the x-axis, we get for $\tan\theta$, where θ is, as before, the angle between the e-ray and the optic axis,

$$\tan\theta=-c^2/a^2m \qquad \dots \quad (10)$$

where m is now given by,

$$m=\frac{\sin 2\alpha\pm 2\sin i\sqrt{c^2\cos^2\alpha+a^2\sin^2\alpha-a^2c^2\sin^2 i}}{2(\cos^2\alpha-a^2\sin^2 i)} \qquad \dots \quad (11)$$

On comparing these two expressions, equations (10) and (11), with the corresponding expressions for negative crystals, equations (5) and (4) respectively, we find that they differ from each other only in a and c being interchanged. In view of the fact that now $\mu_0=1/a$ and $\mu_e=1/c$, and that, as before, it is the positive value of the radical which is cognate to the case in hand, the expression for $\tan\theta$ in terms of the refractive indices remains exactly the same as before.

Thus for all uni-axial crystals, whether positive or negative, we have,

$$\tan\theta = -\cot(\alpha+r') = \frac{-2Q(\mu_0{}^2\cos^2\alpha - \sin^2 i)}{P\sin 2\alpha + 2\sin i\sqrt{R}} \quad \dots \quad (12)$$

And, similarly, the angle between the extra-ordinary wave normal and the optic axis is, in all cases of uni-axials, given by equation (8).

THE USUAL SPECIAL CASES

The various special cases usually given in text books and mentioned before are immediately obtained by putting the appropriate values of α or i in the expressions for θ and ϕ (or r' and r''). These are given below.

Case i.—The optic axis parallel to the refracting surface and in the plane of incidence :

Putting $\alpha=0$ in the expression for $\tan(\alpha+r')$ in equation (7) we get

$$\tan r' = \frac{\mu_e}{\mu_0}\cdot\frac{\sin i}{\sqrt{\mu_0{}^2-\sin^2 i}}\,.$$

But $\sin i=\mu_0 \sin r$, and therefore

$$\mu_0\cos r = \sqrt{\mu_0{}^2-\sin^2 i}.$$

$$\therefore \qquad \frac{\tan r}{\tan r'} = \frac{\mu_0}{\mu_e}.$$

And since $\qquad \tan\phi = \frac{\mu_e{}^2}{\mu_0{}^2}\cdot\tan\theta,$

$$\frac{\tan r}{\tan r''} = \frac{\mu_e}{\mu_0}.$$

Case ii.—The optic axis perpendicular to the refracting surface :

In this case, $\alpha=90°$. Putting this value in equation (7), we get,

$$\tan(90°+r') = -\cot r' = -\frac{\mu_e}{\mu_0}\cdot\frac{\sqrt{\mu_e{}^2-\sin^2 i}}{\sin i}\,,$$

or, $$\tan r' = \frac{\mu_0}{\mu_e}\cdot\frac{\sin i}{\sqrt{\mu_e{}^2-\sin^2 i}}\,.$$

So, $$\tan r'' = \frac{\mu_e}{\mu_0}\cdot\frac{\sin i}{\sqrt{\mu_e{}^2-\sin^2 i}}\,.$$

Case iii.—Normal incidence, i.e., any inclination of the optic axis in the plane incidence, but the angle of incidence is zero :

Putting $i=0$ in the expression for $\tan\theta$ in equation (12), we get

$$\tan\theta = -\cot(\alpha+r') = -\frac{\mu_0}{\mu_e}\cdot\frac{2\mu_0{}^2\cos^2\alpha}{\mu_0\mu_e\sin 2\alpha}\,.$$

Thus,

$$\tan\theta = -Q^2.\cot\alpha \quad \ldots \quad (13)$$

Or,

$$\tan(\alpha + r') = \frac{1}{Q^2}\cot\alpha \quad \ldots \quad (14)$$

Expanding the left hand side of the equation (14) and collecting terms, we get

$$\tan r' = \frac{\mu_e^2 - \mu_0^2}{\mu_e^2\tan\alpha + \mu_0^2\cot\alpha} \quad \ldots \quad (15)$$

For the extra-ordinary wave normal, at normal incidence, we have,

$$\tan\phi = -\cot(\alpha + r'') = \frac{\mu_e^2}{\mu_0^2}\tan\theta = -\cot\alpha,$$

and therefore,

$$r'' = 0 \quad \ldots \quad (16)$$

Peculiarities at Normal Incidence

At normal incidence, the extra-ordinary wave front remains parallel to the incident (plane) wave front and, therefore, the extra-ordinary wave normal does not suffer any refraction, so that $r''=0$, as given by equation (16). The extra-ordinary ray, however, as is well known, does get refracted even at normal incidence and shows some special peculiarities.

Thus, in the first place, since $\tan\theta = -Q^2\cot\alpha$, and since α, whether positive or negative, can lie (numerically) only between 0° and 90°, θ and α must always be of opposite signs. This implies that the refracted ray and the refracting surface must always lie on opposite sides of the optic axis. Further insight into the matter is obtained by examining the equation (15). For negative crystals, μ_0 is greater than μ_e. If, then, α is positive, r' must be negative, and *vice versa*. That is to say that in the case of negative crystals, at normal incidence, the refracted extra-ordinary ray and the optic axis lie on opposite sides of the surface normal. In the case of positive crystals, since now it is μ_e which is greater than μ_0, α and r' will always be of the same sign. In other words, in the case of positive crystals, at normal incidence, the e-ray will lie between the surface normal and the optic axis.

When α is 0° or 90°, then, again it is well known and can be easily deduced from the above expressions for θ or r' (equations (6) and (7) which have been combined together in equation (12), the e-ray also does not suffer any refraction, *i.e.*, $r'=0$. It must, therefore, follow that for some inclination of the optic axis, r' must be a maximum. That this is so, is seen immediately from the curves in Fig. 2, the data for which are given in Table I These have been worked out for calcite, a negative, crystal, and quartz, a positive crystal. For these calculations the refractive indices taken for the two crystals are as given below, Table II, taken from Kaye and Laby's Tables. The angles have been calculated to the nearest minute of arc.

TABLE I

Variation in r' with α (normal incidence)

α	r' (normal incidence) in quartz	r' (normal incidence) in calcite
0°	0°	0°
±10°	±0° 7′	∓1° 56′
±20°	±0° 13	∓3° 42′
±30°	±0° 18′	∓5° 7′
±40°	±0° 20′	∓6° 1′
±50°	±0° 20′	∓6° 15′
±60°	±0° 17′	∓5° 42
±70°	±0° 13′	∓4° 22′
±80°	±0° 7′	∓2° 23′
90°	0°	0°

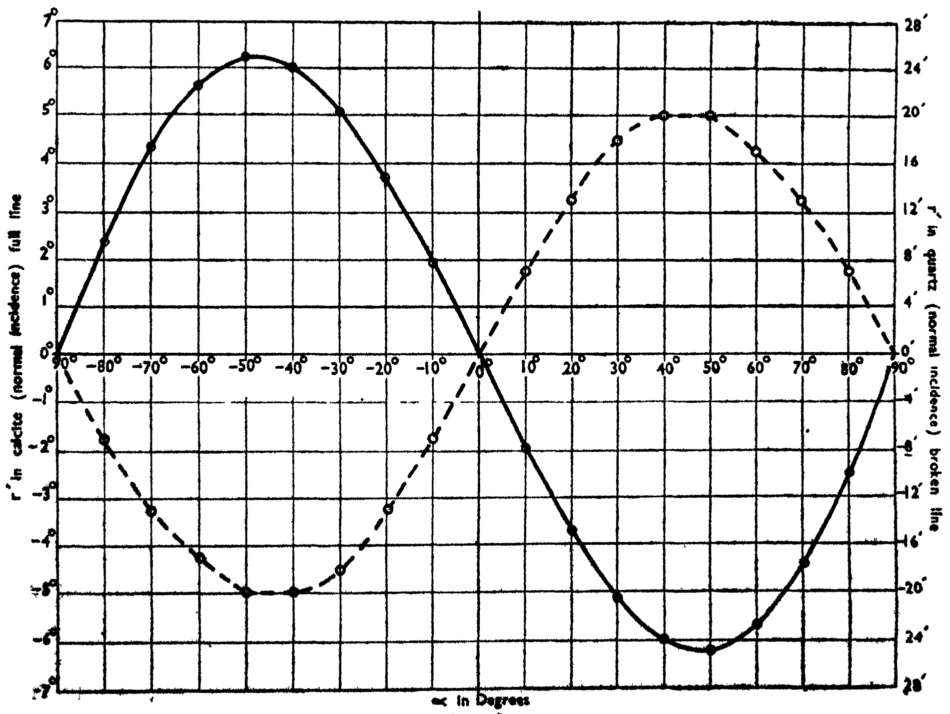


Figure 2

TABLE II

Refractive indices for λ 5893

Crystal	μ_o	μ_e
Calcite (calcspar)	1.6584	1.4864
Quartz	1.5443	1.5534

The exact angle, α, for which r' will be a maximum can be easily calculated in the usual manner. Thus, since $r' = \theta - \alpha \pm 90°$, $dr'/d\alpha = d\theta/d\alpha - 1$. The maximum r' will, therefore, be obtained when $d\theta = d\alpha$ or $\theta = \alpha$. Putting this value of θ in the expression for $\tan\theta$ (equation 13) and remembering that θ and α are always of opposite signs, we get that the angle, α, which corresponds to the maximum r' is given by

$$\tan\alpha = Q = \mu_e/\mu_o.$$

The corresponding value of r' is $2\alpha - 90°$. The values of α for the maximum r' and the corresponding values of r' for the two crystals taken here are given below, Table III.

TABLE III

Maximum r' and the appropriate α

Crystal	α to give maximum r'	Corresponding r'
Quartz	$\pm 44°\ 50'$	$\pm 0°\ 20'$
Calcite	$\pm 48°\ 8'$	$\mp 6°\ 16'$

Oblique Incidence.—The case of normal incidence has been given in detail in the preceding section. For other angles of incidence as well, the angle of refraction of the extraordinary ray passes through a maximum for a certain inclination of the optic axis. It is, however, not so easy to calculate this inclination for oblique incidence as it was in the case of normal incidence. For putting $\theta = \alpha$ in the general expression for $\tan\theta$ (equation 6), we get a cubic in $\cos^2\alpha$ (or $\sin^2\alpha$). That r' does pass through a maximum for any angle of incidence as α is varied from 0° to 90° can, however, be easily seen by calculating r' for various α's and i's. Such an investigation also reveals a few other peculiarities in the behaviour of the extra-ordinary ray, peculiarities not mentioned in the usual text books.

These peculiarities are (*i*) that for certain angles of incidence and certain inclinations of the optic axis, the extra-ordinary ray lies on the same side of the normal as the incident ray; and (*ii*) that under certain conditions the extra-ordinary ray is refracted away from the normal, *i.e.*, the angle of refraction is greater than the angle of incidence. Since the angle of refraction of the extraordinary ray also is, generally, smaller than the angle of incidence, it follows as a corollary that in the circumstances of peculiarity (*iii*) there must be a certain angle of incidence such that the incident and the refracted rays lie along the same straight line, in other words, there is no refraction even at oblique incidence.

It thus turns out that the extra-ordinary ray transgresses the usual laws of refraction in more ways than what one is led to think from the usual statement that the extraordinary ray, in general, does not obey any of the laws of refraction. So far as the very first of these laws is concerned, which says that the refracted ray lies in the plane of incidence, it is obeyed by the extraordinary ray as well in the cases dealt with in this paper, the optic axis having been taken everywhere to lie in the plane of incidence. The second law of refraction, Snell's law, may for the extra-ordinary ray, when the optic axis lies in the plane of incidence, be considered to be replaced by the rather complicated equation (12).

All these peculiarities are seen to a much more pronounced extent in the case of calcite where the difference of the two refractive indices is a little over 0.17 than with quartz where this difference is nearly 19 times smaller being slightly less than 0.01. The tables and the figure of the previous section exemplify this.

With respect to the extra-ordinary wave normal, *i.e.*, the normal to the extra-ordinary wave front, as has already been stated in section 6, most text books do not specifically give its direction even in the usual special cases. Planck (1935), however, has shown by the application of the boundary conditions to the equations of the electro-magnetic fields, that the extra-ordinary wave normal obeys all the laws of refraction even in the most general case of biaxial crystals, any angle and plane of incidence: It always lies in the plane of incidence; and the ratio of $\sin i$ to $\sin r''$ is equal to the refractive index concerned, *i.e.*, the refractive index of the wave in the direction of the wave normal. The first of these laws does not fall within the purview of the present investigation; but the second law, the modified Snell's law, follows from the equation (8), which will be found to be satisfied by the relation, $\sin i = \mu . \sin r''$, where μ is the wave refractive index in the direction which makes an angle of ϕ with the optic axis and is given by the usual relation,

$$\frac{1}{\mu^2} - \frac{1}{\mu_0^2} = \left(\frac{1}{\mu_e^2} - \frac{1}{\mu_0^2}\right) . \sin^2\phi .$$

It should be noticed, however, that the modified Snell's law cannot give the direction of the wave normal which can be determined only with the help of equation (8).

Full data and details about all these special peculiarities mentioned in this section will be discussed in future.

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REFERENCE

Plank, 1935, Treatise on Light, p 193.